Optimal Search for Parameters in Monte Carlo Simulation for Derivative Pricing

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Outline

Introduction

- Derivatives are financial instruments whose values depend on other more fundamental financial assets.
- As derivatives play essential roles in financial markets, pricing them efficiently and accurately is of vital importance.
- Most derivatives are not known to have analytical formulas for their prices.
- Consequently, such derivatives are necessarily priced by numerical methods such as Monte Carlo simulation.

Monte Carlo Simulation

- The standard Monte Carlo simulation has a simple bound of $O(1/\sqrt{N})$ for the standard error for N paths.
- However, when Monte Carlo simulation
  - is combined with other approximation techniques or
  - is used to price complicated financial instruments,

  analytical analysis of its convergence is difficult to obtain.
- So it is essential to have general and efficient algorithms to search for suitable parameter values (e.g., the number of paths) in Monte Carlo simulation that are required to achieve desired precisions while minimizing running time or other computational resources.

Searching Problem

- Searching for an object is of fundamental importance to almost all areas of computer science.
- This paper provides optimal online algorithms:
  - Deterministic algorithm
  - Randomized algorithm

  to search for suitable parameter values in Monte Carlo simulation for derivative pricing which are needed to achieve desired precisions.[1]

[1] The detailed proofs for their competitive ratios and the optimality of the algorithms can be found in the paper.

Monte Carlo Simulation for Derivative Pricing

- Geometric Brownian motion:

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{1}$$

where r is the risk-free rate, σ is the volatility of the stock prices, and the random variable dW_t is the standard Brownian motion.

- Equation (1) has the following solution:

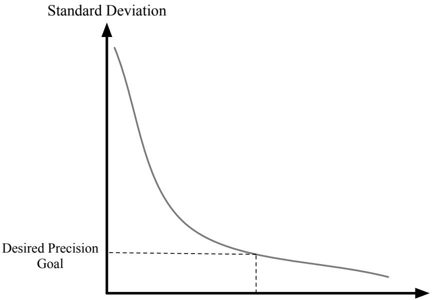
$$S_t = S_0 e^{(r-\sigma^2/2)t+\sigma W_t}. \tag{2}$$

Monte Carlo Simulation for Derivative Pricing (cont.)

- Numerous paths are sampled to obtain the expected payoff in a risk-neutral world.
- Then we discount this payoff at the risk-free rate.
- Pricing of a European-style vanilla call option:
  1. Sample a normal random variable $W_T \sim N(0, T)$ and plug it into Equation (2) to obtain S_T.
  2. Calculate the payoff $\max(S_T - X, 0)$ from the option.
  3. Repeat steps 1 and 2 for d paths to obtain d sample values of the payoff from the option.
  4. Calculate the mean of the sample payoffs to get an unbiased estimate of the expected payoff in a risk-neutral world.
  5. Discount the expected payoff at the risk-free rate to get an estimate of the option price.

Monte Carlo Simulation for Derivative Pricing (concluded)

- The precision of the above estimated price has a bound of $O(1/\sqrt{N})$ (which depends on the number of paths).
  - It is usually measured by the standard error of the estimate.
- For an effective simulation method, the standard error should be a decreasing function of the number of paths.



Problem Formulation

- Given a desired precision goal, the search problem `P` is defined as follows:
    - Find the least number of paths that is required to achieve the given precision.
    - We call this number the `goal` and denote it as N hereafter.

Deterministic Algorithm

- Algorithm `D` is a deterministic geometric sweep algorithm with geometric ratio $r > 1$.

$d \leftarrow 1$;
repeat
  Run the simulation program with d paths and test if the precision of the estimate achieves the desired precision;
  $d \leftarrow d \cdot r$;
until `goal` **achieved**;

Deterministic Algorithm (cont.)

Theorem

For any fixed $r > 1$, algorithm D *has the worst-case competitive ratio*

$$\frac{r^2}{r-1}. \tag{3}$$

Corollary

The unique solution of the equation

$$r^2 - 2r = 0$$

for $r > 1$ is $r^* = 2$, which minimizes Equation (3).

Deterministic Algorithm (concluded)

Lemma

The lower bound for the worst-case competitive ratio for any deterministic algorithm is 4.

Corollary

The deterministic algorithm D *with* $r = 2$ *is optimal.*

Randomized Algorithm

- Algorithm R is a randomized geometric sweep algorithm with geometric ratio $r > 1$.

$\theta \leftarrow$ A random real uniformly chosen from$[0, 1)$;
$d \leftarrow r^{\theta}$;
repeat
    Run the simulation program S with d paths and test if the precision of the estimate achieves the desired precision;
    $d \leftarrow d \cdot r$;
until goal **achieved**;

Randomized Algorithm (cont.)

Theorem

For any fixed $r > 1$, algorithm $\mathtt{R}$ *has competitive ratio*

$$\frac{r}{\ln r}. \tag{4}$$

Corollary

The unique solution of the equation

$$-\frac{1}{(\ln r)^2} + \frac{1}{\ln r} = 0$$

for $r > 1$ is $r^* = e$, which minimizes Equation (4).

Randomized Algorithm (concluded)

Theorem

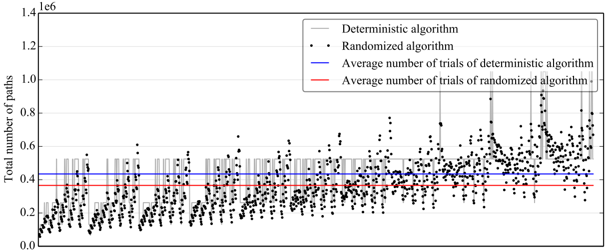
The optimal competitive ratio is given by

$$\min_{r>1} \left\{ \frac{r}{\ln r} \right\}. \tag{5}$$

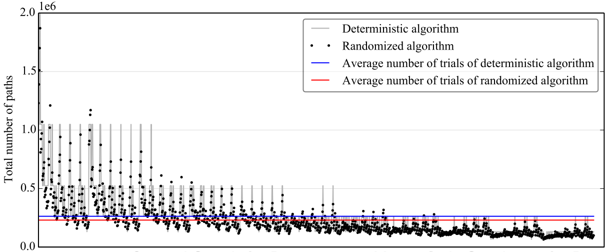
Since this ratio is achieved by algorithm R, *algorithm* R *is optimal.*

Results of European Options

(a) Vanilla Call Options.



(b) Vanilla Put Options.

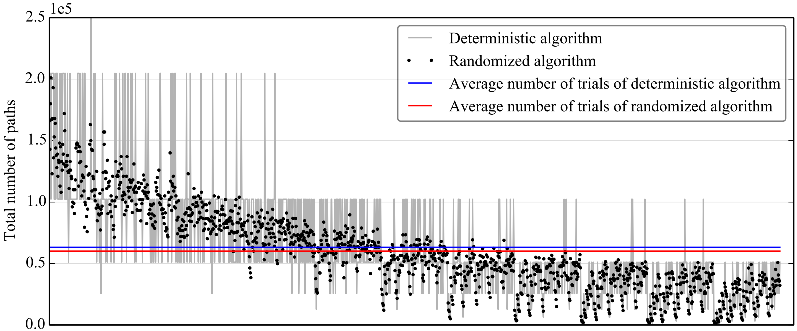


Pricing of American Options

- One might ask “why can’t we just compute standard error after sampling each additional path?”
- Indeed, we do not need to throw away the prices from the already performed paths because each additional sample would not affect the prices generated by other paths.
- However, for most of complex derivatives pricing problems (especially in dealing with American-style derivatives), computing sample standard error after sampling each additional path is not applicable because sampling additional one path would result in recalculating the prices of all other paths.
- The Least-squared Monte Carlo technique: American-style options
  - Each additional path results in recalculating the prices of all other paths.
  - Every path is included when identifying the conditional expected value of continuation via regression.

Results of American Options

American Put Options Priced by Least-Squared Monte Carlo Simulation



Conclusions

- This paper provides an optimal deterministic online algorithm and an optimal randomized online algorithm to search for desired parameter values in Monte Carlo simulation for derivative pricing.
- Our experiments on both European-style and American-style options show that our proposed approach can efficiently and effectively determine a suitable number of paths for pricing these options with Monte Carlo simulation within a desired precision.
- Further research: Identify other complex derivatives for which analytical analysis of their convergence are difficult to obtain and computing the standard error after sampling each additional path is not applicable.