On the Complexity of Bivariate Lattice with Stochastic Interest Rate Models

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Outline

Introduction

- The pricing of financial instruments with two or more state variables has been intensively studied.
- The added state variables besides the stock price can be volatility or interest rate.
- This paper studies bivariate lattices with a stock price component and an interest rate component.
- It can be used to price interest rate-sensitive securities such as callable bonds and convertible bonds (CBs).

Interest Rate Models

- This paper focuses on lognormal interest rate models such as the Black-Derman-Toy (BDT), Black-Karasinski, and Dothan models.
- In particular, this paper adopts the popular BDT model to explain the main ideas of our bivariate lattice.
  - Our techniques work for all short rate models.

Related Work

- Using the BDT model, Hung and Wang (2002) propose a bivariate binomial lattice to price CBs.
- Chambers and Lu (2007) extend it by including correlation between stock price and interest rate.
- The lattices' sizes are both cubic in the total number of time steps.
- Unfortunately, this paper shows that both works share a serious flaw: invalid transition probabilities.
    - Their lattices cannot grow beyond a certain time without encountering invalid transition probabilities.

Main Results

- This paper proposes the first bivariate lattice that guarantees valid transition probabilities even when interest rates can grow without bounds.
- Our bivariate lattice has two components: stock price and interest rate.
  - The interest rate component: a binomial interest rate lattice for the BDT model.
  - The stock price component: a trinomial lattice with mean tracking.
- We then combine both lattices in such a way that
  (1) The bivariate lattice is free of invalid transition probabilities;
  (2) The bivariate lattice grows superpolynomially if the interest rate model allows rates to grow superpolynomially (such as the BDT model);
  (3) The above bound is optimal.

Main Results (cont.)

- Two popular beliefs:
  (1) It is routine to build a bivariate lattice from a lattice for stock price and a lattice for interest rate.
  (2) The resulting bivariate lattice is of polynomial size when its lattice components are.

Main Results (cont.)

- Two findings in the paper contradict the popular beliefs.
  (1) The resulting bivariate lattice by the popular method of combining two individual lattices is usually invalid.
  (2) The bivariate lattice for stock price and interest rate grows *superpolynomially* if the interest rate model allows rates to grow superpolynomially such as lognormal models.

Modeling and Definitions

- The stock price follows a geometric Brownian motion with a constant volatility σ and a constant riskless rate r:

$$\frac{dS}{S} = rdt + \sigma dz,$$

  where dz denotes a standard Brownian motion.

- In the BDT model, the short rate r follows the stochastic process,

$$d \ln r = \theta(t)dt + \sigma_{\mathrm{r}}(t)dz,$$

  where

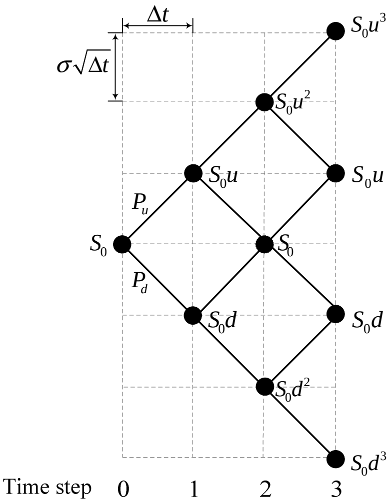
  - $\theta(t)$ is a function of time that makes the model fit the market term structure;
  - $\sigma_{\mathrm{r}}(t)$ is a function of time and denotes the instantaneous standard deviation of the short rate;
  - dz is a standard Brownian motion.

The Binomial Lattice Model

- The size of one time step is $\Delta t = T/n$.
- u, d, P_u, P_d:
  - Match the mean and variance of the stock return asymptotically.
  - $ud = 1$.
  - $P_u + P_d = 1$.
- A solution is:

$$u = e^{\sigma\sqrt{\Delta t}}, P_u = \frac{e^{r\Delta t} - d}{u - d},$$

$$d = e^{-\sigma\sqrt{\Delta t}}, P_d = \frac{e^{r\Delta t} - u}{d - u}.$$



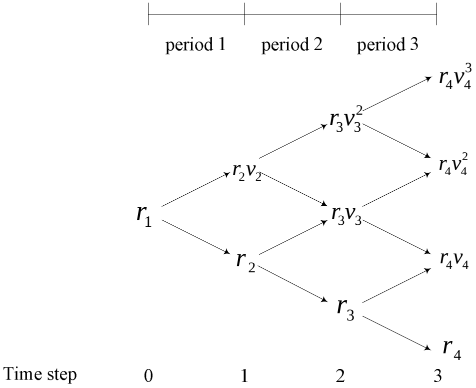
The BDT Binomial Interest Rate Lattice

- A binomial lattice consistent with the term structures.
- The probability for each branch is 1/2.
- There are j possible rates (which are applicable to period j) at time step $j-1$:

$$r_j, r_j v_j, r_j v_j^2, \ldots, r_j v_j^{j-1},$$

where

$$v_j = e^{2\sigma_j \sqrt{\Delta t}}.$$



The Invalid Transition Probability Problem

- Assume no correlation between stock price and interest rate.
- The no-arbitrage requirements $0 \leq P_u, P_d \leq 1$ are equivalent to

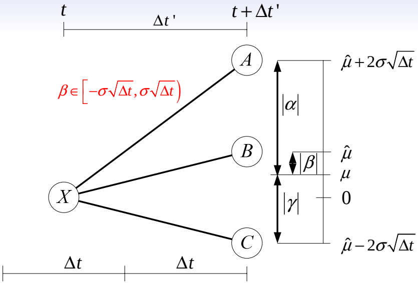
$$d < e^{r\Delta t} < u. \tag{1}$$

- It is known that $E[S_{t+\Delta t}/S_t] = e^{r\Delta t}$.
- Inequalities (1) say the top and bottom branches of a node at time t must bracket the mean stock price of the next time step, at time $t + \Delta t$.
- Note that $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$ are independent of r, when the maximum r grows without bounds such as the BDT model.
- Inequalities (1) will break eventually.

A Valid Bivariate Lattice

- The bivariate lattice has two components: stock price and interest rate.
    - The interest rate component will follow the BDT binomial lattice.
    - The stock price component will be the trinomial lattice with the nodes placed as the binomial lattice.
- To guarantee valid transition probabilities, the top and bottom branches from every node must bracket the mean stock return.

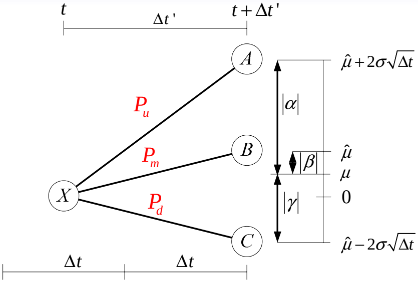
Trinomial Structure



The branching probabilities for the node X

$$
\begin{aligned}
\beta &\equiv \hat{\mu} - \mu, \\
\alpha &\equiv \hat{\mu} + 2\sigma\sqrt{\Delta t} - \mu = \beta + 2\sigma\sqrt{\Delta t}, \\
\gamma &\equiv \hat{\mu} - 2\sigma\sqrt{\Delta t} - \mu = \beta - 2\sigma\sqrt{\Delta t}, \\
\hat{\mu} &\equiv \ln\left(s(B)/s(X)\right).
\end{aligned}
$$

Trinomial Structure (concluded)

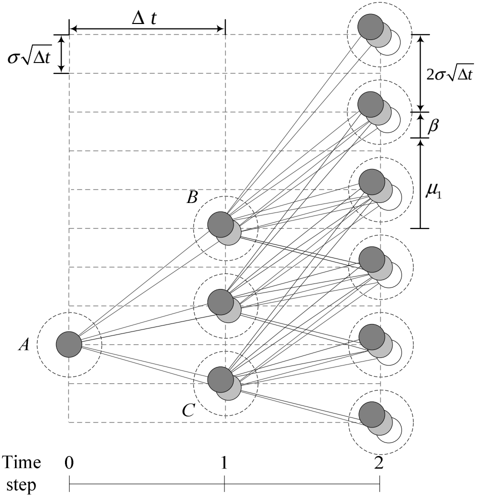


The branching probabilities for the node X

$$
\begin{aligned}
P_u \alpha + P_m \beta + P_d \gamma &= 0, \\
P_u (\alpha)^2 + P_m (\beta)^2 + P_d (\gamma)^2 &= \text{Var}, \\
P_u + P_m + P_d &= 1.
\end{aligned}
$$

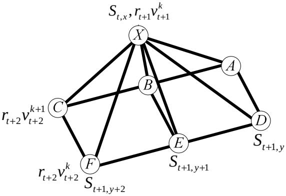
A 2-Period Mean-Tracking Trinomial Lattice

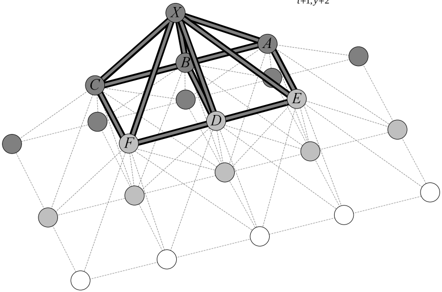
- Let $d(\ell)$ denote the number of stock prices spanned by the highest stock price and the lowest one at time step ℓ.
  - For instance, on the right, $d(0) = 1$, $d(1) = 3$, and $d(2) = 6$.



The Bivariate Lattice

- Each node in a bivariate lattice corresponds to a bivariate state with a stock price and an interest rate.
- There are $2 \times 3 = 6$ branches per node.
  - Node X at time step t has 6 branches, to nodes A, B, C, D, E, and F at time step $t+1$.





Complexity of Bivariate Lattices

- For a binomial interest rate lattice, there are $j+1$ nodes at time step j.
- Suppose there are $d(j)$ nodes at time step j on the trinomial lattice for the stock price.
- Then the bivariate lattice has $(j+1)d(j)$ nodes at time step j.
- Therefore, the total node count of the bivariate lattice is

$$\sum_{j=0}^{n}(j+1)d(j).$$

Complexity of Bivariate Lattices (cont.)

- Fix the maturity $T = 1$ for simplicity.
- Let μ_{j-1} denote the mean of the logarithmic stock return one time step from the nodes with the largest interest rate at time step $j-1$, i.e.,

$$\mu_{j-1} = r_j v_j^{j-1} - \frac{\sigma^2}{2n} = r_j e^{2(j-1)\sigma_j/\sqrt{n}} - \frac{\sigma^2}{2n}$$

  for $j = 1, 2, \ldots, n+1$.
- The previous figure shows that $d(0) = 1$, $d(1) = 3$, and

$$d(2) = d(1) + 1 + \frac{\mu_1 + \beta + 2\sigma/\sqrt{n} - \sigma/\sqrt{n}}{2\sigma/\sqrt{n}}.$$

- As $\beta \in [-\sigma\sqrt{\Delta t}, \sigma\sqrt{\Delta t})$, we know that

$$d(2) \leq d(1) + 2 + \frac{\mu_1}{2\sigma/\sqrt{n}} = d(1) + 2 + n^{0.5}\,\frac{\mu_1}{2\sigma}.$$

Complexity of Bivariate Lattices (cont.)

- Inductively,

$$d(j) \leq d(1) + 2(j-1) + n^{0.5} \sum_{k=1}^{j-1} \frac{\mu_k}{2\sigma} = 1 + 2j + n^{0.5} \sum_{k=1}^{j-1} \frac{\mu_k}{2\sigma}. \quad (2)$$

- If the interest rate model allows rates to grow superpolynomially such as the BDT model, μ_j will grow superpolynomially in magnitude.
- The $d(j)$ in turn grows superpolynomially.
- As a result, the size of our bivariate lattice grows superpolynomially.

Conclusions

- This paper presents the first bivariate lattice to solve the invalid transition probability problem even if the interest rate model allows rates to grow superpolynomially in magnitude.
- We prove that the bivariate lattice method for stock price and interest rate grows superpolynomially if
  1. The transition probabilities are guaranteed to be valid and
  2. The interest rate model allows rates to grow superpolynomially such as the BDT model.
- In the process, we have shown that the common way of constructing bivariate lattices from univariate lattices is incorrect.
- Our lattice construction is optimal if the interest rate component is the BDT model.