

A Closed-form Formula for an Option with Discrete and Continuous Barriers

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Introduction

- Barrier options are derivatives that are knocked in or knocked out when the price of the underlying asset reaches a specific level during the life of the options.
- These types of contracts are popular because they are less expensive than the identical vanilla options and provide additional flexibility.

Related Work

- Merton (1973) first proposes a closed-form solution for a European down-and-out call.
- Rubinstein and Reiner (1991), Benson and Daniel (1991), and Hudson (1992) provide further extensions.
- Carr (1995) develops closed-form solutions for two extensions of European up-and-out call options: protected barrier options and rainbow barrier options.
- Barrier options with discrete monitoring dates have been studied in, for example, Broadie et al. (1997) and Fusai et al. (2006).
- However, little has been published on the analytical solutions for derivatives with both continuously and discretely monitored barriers.

Main Results

- This paper presents a methodology to derive closed-form solutions for a class of derivative products with one continuously and a few discretely monitored barriers.
- This class of structured products is called knock-out double-income (KODI).
 - A concrete one was issued by Taiwan's Polaris Securities in 2004.

The Economic Rationale of KODI

- In 2004, the technology sector of Taipei's stock market declined substantially in market value, and investors suffered heavy losses of approximately 20%.
- A typical investor behavior would be to sell the stock the moment it regains the initial purchase price or hold the stock if their market value continues to slide.
- Polaris advised their clients to sell off the stock, and buy a product from the KODI class linked to the same stock.

The Economic Rationale of KODI (cont.)

- The KODI is a structured product with a one-year tenor and 4 monitoring dates, once every quarter.
- The product has three distinctive features.
 - First, the holder will obtain a 120% rebate when the stock exceeds 110% of the initial price at each monitoring date (the double-income part).
 - Second, the stock will be returned to the holder as the stock price hits the continuous low barrier, which is 85% of the initial price.
 - Finally, if this KODI does not mature early, the holder will be paid the initial price at maturity if the stock price is less than the initial stock price (the principal guarantee feature).

The Economic Rationale of KODI (cont.)

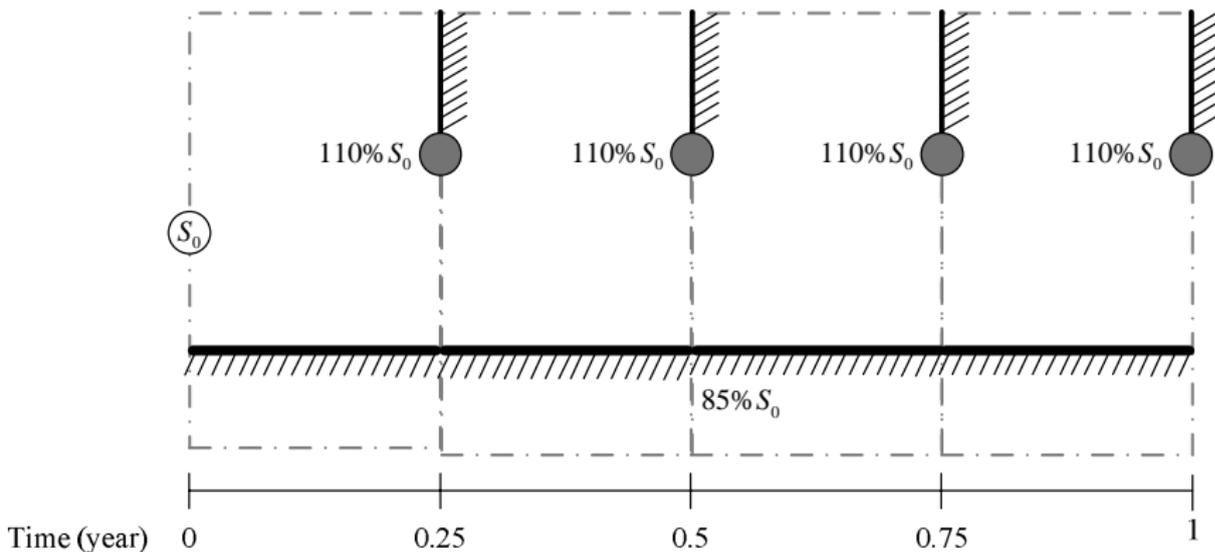


Figure: Graphical Representation of KODI.

Valuation of KODI

- The stock price follows a geometric Brownian motion with a constant volatility σ , constant riskless rate r , and constant dividend yield q :

$$dS_t = (r - q) S_t dt + \sigma S_t dW_t,$$

where W_t denotes a standard Brownian motion under the risk-neutral measure P .

Valuation of KODI (cont.)

- We examine a product from the KODI class with maturity T and 3 monitoring dates t_1 , t_2 , and $t_3 = T$ for simplicity.
- At each monitoring date, if the price of the stock exceeds the discrete high barrier H , KODI will mature early and the holder will receive the rebate R .
- On the other hand, if at any time the price of the stock hits below the continuous low barrier L , KODI will mature early and the underlying stock will be returned to the holder.
- If KODI does not mature early, there are two cases at maturity.
 - If the stock price S_T is greater than the initial price S_0 , the payoff will be S_T .
 - Otherwise, the client will be paid the initial price S_0 .

Valuation of KODI (cont.)

- We first normalize the variables as follows.

- $X_t \equiv \ln(S_t/S_0)$.
- By Ito's lemma,

$$dX_t = \mu dt + \sigma dW_t,$$

where $\mu = r - q - \sigma^2/2$.

- Moreover, let $h \equiv \ln(H/S_0)$ and $\ell \equiv \ln(L/S_0)$.
- Define $m_{ab} = \min_{u \in [t_a, t_b]} X_u$ and denote $\mathbf{1}\{A\}$ as the indicator function of the event A .

Valuation of KODI (cont.)

- The value of our KODI with 3 monitoring dates (U) can be decomposed into three parts:

$$U = A + B + C.$$

- The first type of expectations, Eqs. (1), (2), and (3), deals with the events that the KODI knocks out because of hitting the low barrier ℓ .

$$A = E[e^{-r\tau} S_\tau \mathbf{1}\{m_{01} < \ell\}] \quad (1)$$

$$+ E[e^{-r\tau} S_\tau \mathbf{1}\{m_{01} > \ell, X_1 < h, m_{12} < \ell\}] \quad (2)$$

$$+ E[e^{-r\tau} S_\tau \mathbf{1}\{m_{01} > \ell, X_1 < h, m_{12} > \ell, X_2 < h, m_{23} < \ell\}] \quad (3)$$

- Define $\tilde{\mu} = \sqrt{\mu^2 + 2\sigma^2 r}$.
- The first type of expectations can be evaluated as

$$E[e^{-r\tau} \cdot \mathbf{1}\{\text{event}\}] = e^{\ell(\mu - \tilde{\mu})/\sigma^2} \tilde{E}[\mathbf{1}\{\text{event}\}] = e^{\ell(\mu - \tilde{\mu})/\sigma^2} \tilde{P}(\text{event}),$$

where the drift of the process X is $\tilde{\mu}$ under the probability measure \tilde{P} .

Valuation of KODI (cont.)

- The second type of expectations handles the cases with a constant rebate where the KODI does not knock out in the previous stages and the stock price falls within a certain range at each monitoring date:

$$B = E \left[e^{-rt_1} R \mathbf{1}\{m_{01} > l, X_1 > h\} \right] \quad (4)$$

$$+ E \left[e^{-rt_2} R \mathbf{1}\{m_{01} > l, X_1 < h, m_{12} > l, X_2 > h\} \right] \quad (5)$$

$$+ E \left[e^{-rt_3} R \mathbf{1}\{m_{01} > l, X_1 < h, m_{12} > l, X_2 < h, m_{23} > l, X_3 > h\} \right] \quad (6)$$

$$+ E \left[e^{-rt_3} S_0 \mathbf{1}\{m_{01} > l, X_1 < h, m_{12} > l, X_2 < h, m_{23} > l, X_3 < X_0\} \right] \quad (7)$$

- Eqs. (4), (5), and (6) are for the events that the KODI knocks out because of hitting the high barrier h , and
- Eq. (7) handles the principal guarantee feature at maturity when the KODI does not mature early and $X_3 < X_0$.

Valuation of KODI (cont.)

- For the second type of expectations, the identity,

$$E[e^{-rt_i} R \cdot \mathbf{1}\{\text{event}\}] = e^{-rt_i} RP(\text{event}),$$

can be applied to Eqs. (4), (5), and (6) with $i = 1, 2, 3$, respectively, whereas

$$E[e^{-rt_3} S_0 \cdot I\{\text{event}\}] = e^{-rt_3} S_0 P(\text{event})$$

can be applied to Eq. (7).

Valuation of KODI (cont.)

- Equation (8) is of the third type, related to the events that the KODI does not knock out but $X_0 < X_3 < h$:

$$C = E \left[e^{-rt_3} S_{t_3} \cdot I\{m_{01} > l, X_1 < h, m_{12} > l, X_2 < h, m_{23} > l, X_0 < X_3 < h\} \right] \quad (8)$$

- For the third type of expectation, Eq. (8), we define $\mu^* = \mu + \sigma^2/2 = r - q + \sigma^2/2$ and obtain

$$E[e^{-rt_3} S_{t_3} \cdot I\{\text{event}\}] = S_0 E^*[I\{\text{event}\}] = S_0 P^*(\text{event}),$$

where the drift of the process X is μ^* under the probability measure P^* .

- As $\tilde{P}(\text{event})$ and $P^*(\text{event})$ can be directly obtained by replacing μ in $P(\text{event})$ with $\tilde{\mu}$ and μ^* , respectively, we shall cover how to derive the probabilities under the risk-neutral probability measure P only.

Valuation of KODI (cont.)

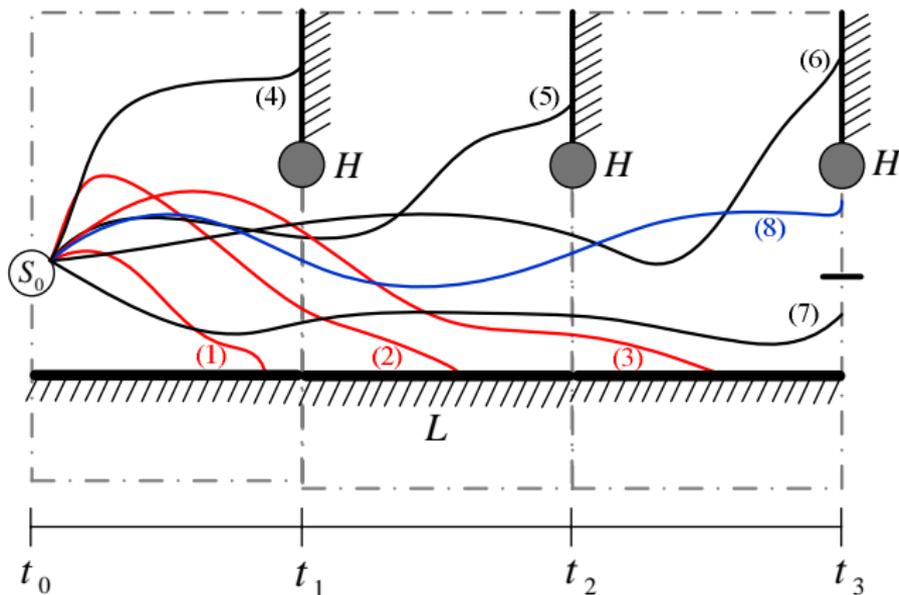


Figure: Three Types of Expectations.

An Example

- We take the derivation of $P(m_{01} > \ell, X_1 < h, m_{12} < \ell)$ (the probability in Eq. (2)) an example.
- First, we have

$$P(m_{01} < \ell) = N\left(\frac{\ell - \mu t_1}{\sigma \sqrt{t_1}}\right) + e^{2\mu\ell/\sigma^2} N\left(\frac{\ell + \mu t_1}{\sigma \sqrt{t_1}}\right) \quad (9)$$

and

$$P(m_{01} > \ell, X_1 > h) = N\left(\frac{-h + \mu t_1}{\sigma \sqrt{t_1}}\right) - e^{2\mu\ell/\sigma^2} N\left(\frac{-h + 2\ell + \mu t_1}{\sigma \sqrt{t_1}}\right). \quad (10)$$

- By Eq. (9) and the stationary-increment property of Brownian motion,

$$\begin{aligned} P(m_{12} < \ell | X_1 = x) &= N\left(\frac{\ell - x - \mu(t_2 - t_1)}{\sigma \sqrt{t_2 - t_1}}\right) \\ &+ e^{2\mu(\ell - x)/\sigma^2} N\left(\frac{\ell - x + \mu(t_2 - t_1)}{\sigma \sqrt{t_2 - t_1}}\right). \quad (11) \end{aligned}$$

An Example (cont.)

- Now, by Eq. (10),

$$P(m_{01} > \ell, X_1 < x) = P(m_{01} > \ell) - N\left(\frac{-x + \mu t_1}{\sigma \sqrt{t_1}}\right) + e^{2\mu\ell/\sigma^2} N\left(\frac{2\ell - x + \mu t_1}{\sigma \sqrt{t_1}}\right).$$

- Define $p(x) \equiv dP(m_{01} > \ell, X_1 < x)/dx$. Then,

$$p(x) = \frac{1}{\sigma \sqrt{t_1}} n\left(\frac{-x + \mu t_1}{\sigma \sqrt{t_1}}\right) - \frac{e^{2\mu\ell/\sigma^2}}{\sigma \sqrt{t_1}} n\left(\frac{2\ell - x + \mu t_1}{\sigma \sqrt{t_1}}\right). \quad (12)$$

Therefore,

$$\begin{aligned} & P(m_{01} > \ell, X_1 < h, m_{12} < \ell) \\ &= \int_{\ell}^h P(m_{12} < \ell | X_1 = x) dP(m_{01} > \ell, X_1 < x) \\ &= \int_{\ell}^h \left(N\left(\frac{\ell - x - \mu(t_2 - t_1)}{\sigma \sqrt{t_2 - t_1}}\right) + e^{2\mu(\ell - x)/\sigma^2} N\left(\frac{\ell - x + \mu(t_2 - t_1)}{\sigma \sqrt{t_2 - t_1}}\right) \right) \\ & \quad \cdot \left(\frac{1}{\sigma \sqrt{t_1}} n\left(\frac{-x + \mu t_1}{\sigma \sqrt{t_1}}\right) - \frac{e^{2\mu\ell/\sigma^2}}{\sigma \sqrt{t_1}} n\left(\frac{2\ell - x + \mu t_1}{\sigma \sqrt{t_1}}\right) \right) dx. \quad (13) \end{aligned}$$

An Example (cont.)

- We will need the reduction formula of Curnow and Dunnett (1962), which expresses the n -dimensional multinormal CDF as the integral of a normal PDF multiplied by an $(n - 1)$ -dimensional multinormal CDF.
- In particular, the 2-dimensional case of the reduction formula is

$$\int_{-\infty}^a n(w) N(\alpha + \beta w) dw = N_2 \left(\left\{ a, \frac{\alpha}{\sqrt{1 + \beta^2}} \right\}; \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right), \quad (14)$$

where $\rho = \frac{-\beta}{\sqrt{1 + \beta^2}}$.

Valuation of KODI

KODI Values and CPU Times						
Formula			Simulation			
		Value	Time (sec)	Value	Time (sec)	Error (%)
Base case		1.00061	2.70	1.00112	376.95	-0.051
S_0	0.95	0.94814	2.64	0.94304	377.36	0.541
	1.05	1.05168	2.66	1.06141	376.89	-0.917
σ	10%	1.00868	2.84	1.00908	377.02	-0.040
	30%	0.98553	2.61	0.98547	377.39	0.006
r	1%	1.00061	2.67	1.00112	376.83	-0.051
	4%	1.00054	2.70	1.00105	376.75	-0.051
q	3%	1.00563	2.67	1.00614	376.95	-0.050
	5%	0.99561	2.69	0.99601	376.86	-0.041
MAPE (%)			0.22548	RMSPE (%)		0.37478

Table: The Values and CPU Times of Pricing Our KODI.

Conclusion

- This paper presents a methodology to derive closed-form solutions for a class of derivative products with one continuously and a few discretely monitored barriers.
- The methodology can be extended to price the products with complex barrier structures.
- The analytical results may inspire the design of new exotic options.