A Novel Tree Model for Evaluating Corporate Debts with Complex Liability Structures and Debt Covenants

Chuan-Ju Wang

Department of Computer Science & Information Engineering
National Taiwan University
Joint work with Prof. Tian-Shyr Dai and Prof. Yuh-Dauh Lyuu

June 30, 2010
Asian Finance Association 2010 Conference
Introduction

- A default-free bond can be priced independently from other outstanding bonds of the same issuer.
- A risky bond, however, cannot be evaluated independently from other outstanding bonds of the same issuer.
- This is because they may change the financial status of the issuer and, as a result, the likelihood of default.
Credit Models

- There are two kinds of credit models: the reduced-form model and the structural model.

- This paper will focus on the structural model.
  - It models the evolution of the firm's asset value and specifies the conditions leading to default.
Structural Credit Models

- Merton (1974)
  - Default can only occur at the bond’s maturity date when the firm’s asset value cannot meet its payment obligations.

- Black and Cox (1976)
  - The firm defaults once its asset value hits an exogenous default boundary.

- The collection of a firm’s outstanding bonds constitutes its liability structure.

- Most of the literature focuses on a liability structure consisting only of a single bond.

- It is difficult to extend their analytical results under more general liability structures.
Bond covenants also affect bond prices.

Common bond covenants include restrictions on asset sales, exogenous default boundaries, seniorities of bonds, and early redemption.
Assumptions on Asset Sales

- The values of risky bonds strongly depend on the assumptions regarding asset sales.

- The no-asset-sales assumption:¹
  - The equity holders are not allowed to sell the firm’s assets to finance the coupon, bond, or dividend payouts.
  - Thus the equity holders have to finance the payments by issuing new equities.

¹Leland (1994).
But allowing asset sales is more common in the real world.

The proportional-asset-sales assumption:²
- The firm is allowed to sell a proportion of the firm’s asset value.
  - If the said proportion of the firm’s asset value is less than the payout, the equity holders will try to finance the shortfall by selling additional equities.

The total-asset-sales assumption:³
- The firm is allowed to finance the total payout by selling the firm’s asset.
- This assumption significantly increases the analytical complexity.

³Brennan and Schwartz (1978).
Default Events

- The default event is triggered once the firm’s asset value hits a default boundary.

- The boundary at time $t$ can be exogenously specified as a function of the firm’s outstanding bonds at time $t$.
  - A constant proportion of the sum of the outstanding bonds’ face values.\(^4\)
  - The discounted present value of the outstanding bonds.\(^5\)

- The default boundary can also be endogenously defined.
  - The firm fails to raise sufficient equity capital to meet current bond obligations.\(^6\)

---

\(^5\) Black and Cox (1976), Briys and De Varenne (1997).
\(^6\) Leland (1994).
Pricing Risky Bonds

- Pricing a risky bond in the presence of other outstanding bonds seems to be first studied by Geske (1977).

- Under the no-asset-sales assumption, the equity value and the longest-term bond can be priced as compound options.
  - Geske (1977) assumes that there is no exogenous default boundary.

- However, it is difficult to extend this method under different bond covenants or assumptions on asset sales.
This paper proposes a flexible lattice for pricing risky bonds with general liability structures and bond covenants.

To model the jumps in the firm’s asset value because of the coupon or bond repayment, we adopt the trinomial structure of Dai and Lyuu (2010).

Our lattice has the flexibility to eliminate the price oscillations by making certain nodes or price levels on the lattice align with the exogenous default boundaries.

Our lattice can deal with the endogenous default boundary, the early redemption, and seniority of bonds.

Finally, our lattice can deal with the jump-diffusion process.
The Dynamics of the Firm’s Asset Value

- Denote the firm’s asset value at time $t$ as $V_t$.
- The firm’s asset value follows the jump-diffusion process.\footnote{Zhou (2001).}

$$dV_t = \left((r - \lambda \bar{k}) V_t - P\right) dt + \sigma V_t dz + k V_t dq,$$

where

- $r$ is the risk-free rate;
- $P$ denotes the payout from selling the firm’s asset per annum to finance bond payouts;
- $\sigma$ denotes the volatility contributed by the diffusion component;
- $dz$ is a standard Brownian motion;
- $k$ denotes the magnitude of the random jump;
- $q$ denotes a Poisson process with an intensity $\lambda$. 
The size of one time step is $\Delta t = T/n$. 

$u, d, P_u, P_d$: 
- Match the mean and variance of the return asymptotically. 
- $ud = 1$. 
- $P_u + P_d = 1$. 

The trinomial structure is used to deal with the jumps in a firm’s asset value.
Lattice Structures under the Lognormal Diffusion Process

- The size of one time step is $\Delta t = T/n$.
- $u, d, P_u, P_d$:
  - Match the mean and variance of the return asymptotically.
  - $ud = 1$.
  - $P_u + P_d = 1$.
- The trinomial structure is used to deal with the jumps in a firm's asset value.
Price Oscillation Problem

- Price oscillation problem is mainly due to the nonlinearity error.
  - Introduced by the nonlinearity of the contingent claim’s value function.

- The solution of the nonlinearity error:
  - Making a node or a price level of the lattice coincide with the critical locations where the value function of the contingent claim is highly nonlinear.

- For the structural model, critical locations occur along the exogenous default boundary and at the time points when bond payouts occur.
The branching probabilities for the node $X$

$$P_u \alpha + P_m \beta + P_d \gamma = 0,$$
$$P_u (\alpha)^2 + P_m (\beta)^2 + P_d (\gamma)^2 = \text{Var},$$
$$P_u + P_m + P_d = 1.$$
Two zero-coupon bonds (face value, maturity date):

- $(F_1, T/2)$
- $(F_2, T)$.

The bond repayments are fully financed by selling the firm’s asset.

The default boundary:

- $\kappa (F_1 + F_2)$ for $[0, T/2]$.
- $\kappa F_2$ for $(T/2, T]$. 

A Novel Tree Model for Evaluating Corporate Debts with Complex Liability Structures and Debt Covenants
Extensions to Other Bond Covenants

- Incorporate different assumptions regarding asset sales.
- $P_1$ and $P_2$ denote the payouts financed by selling the firm’s asset.
  - The proportional-asset-sales assumption of selling a fixed proportion $D$ of the firm's asset value:
    - $P_1 \equiv D \ V_u \Delta t$ and $P_2 \equiv D \ V_d \Delta t$.
  - The no-asset-sales assumption:
    - $P_1 \equiv P_2 \equiv 0$. 

A Novel Tree Model for Evaluating Corporate Debts with Complex Liability Structures and Debt Covenants
Assume a time-varying exogenous default boundary.

The gray nodes are on the curve.

Other nodes are laid out from the gray nodes upward.

Thus the successor nodes of node $X$ will be selected from the nodes at time step 1.

The procedure can be repeatedly applied in the construction of the lattice.
Robustness and Generality

<table>
<thead>
<tr>
<th>$\sigma_V$</th>
<th>Merton (1974)</th>
<th>Black and Cox (1976)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Lattice</td>
<td>Formula</td>
</tr>
<tr>
<td>0.25</td>
<td>2934.82</td>
<td>2934.82</td>
</tr>
<tr>
<td></td>
<td>(0.00003%)</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>2875.60</td>
<td>2875.60</td>
</tr>
<tr>
<td></td>
<td>(0.00014%)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma_V$</th>
<th>Leland (1994)</th>
<th>Geske (1977)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Lattice</td>
<td>Formula</td>
</tr>
<tr>
<td>0.25</td>
<td>3419.57</td>
<td>3419.38</td>
</tr>
<tr>
<td></td>
<td>(−0.00556%)</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>2941.41</td>
<td>2942.23</td>
</tr>
<tr>
<td></td>
<td>(−0.02788%)</td>
<td></td>
</tr>
</tbody>
</table>

**Table:** Accuracy of Our Lattice.
General Liability Structures

<table>
<thead>
<tr>
<th>Maturity of $B_1$</th>
<th>$B_2 \prec B_1$</th>
<th>$B_1 \prec B_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Formula</td>
<td>Lattice</td>
</tr>
<tr>
<td>2.5</td>
<td>475.59 (M)</td>
<td>475.59</td>
</tr>
<tr>
<td>3</td>
<td>470.80 (L)</td>
<td>470.80</td>
</tr>
<tr>
<td>3.5</td>
<td>$\times$</td>
<td>466.12</td>
</tr>
</tbody>
</table>

Table: Pricing Unprotected Bonds under the No-Asset-Sales Assumption.

- **M**: the formula of Merton (1974).
- **G**: the formula of Geske (1977).
- **L**: the formula of Lando (2004).
The credit spreads (bps) of bond $B_1$

<table>
<thead>
<tr>
<th>Total-Asset-Sales</th>
<th>$B_2 \prec B_1$</th>
<th>$B_1 \prec B_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity of $B_1$</td>
<td>Non-putable</td>
<td>Putable</td>
</tr>
<tr>
<td>2.99</td>
<td>0.00002</td>
<td>0.000002</td>
</tr>
<tr>
<td>3</td>
<td>0.00189</td>
<td>0.00189</td>
</tr>
<tr>
<td>3.01</td>
<td>40.86288</td>
<td>0.02893</td>
</tr>
</tbody>
</table>

**Table:** Impacts of General Liability Structures and Bond Covenants on Protected Bonds.
Conclusions

- This paper prices risky bonds by incorporating general liability structures and bond covenants into the structural model.

- An efficient lattice is then presented to price these bonds.

- This accurate numerical method is of great help to explore how credit spreads are influenced by the bond covenants and the change in the liability structure due to bond repayments.

- Furthermore, our lattice can be extended to deal with jump-diffusion processes.

- The numerical results confirm the robustness and generality of our lattice.

- The also show its ability to accurately evaluate the risky bonds with general liability structures and bond covenants.