An Efficient and Accurate Lattice for Pricing Derivatives under a Jump-Diffusion Process

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Introduction

- Pricing derivatives is equivalent to computing its expected payoff under a suitable probability measure.
- Most derivatives have no analytical formulas.
- So they must be priced by numerical methods like the lattice model.
However, the nonlinearity error may cause the pricing results to converge slowly and oscillate significantly.¹

¹Figlewski and Gao (1999).
Models

- Lognormal diffusion process has been widely used to model the stock price dynamics but is incapable of capturing empirical stock price behaviors.\(^2\)

- Many alternative processes like jump-diffusion process have been proposed to address this problem.

Amin (1993)
- He approximates the jump-diffusion process by a multinomial lattice.
- Huge numbers of branches at each node make the lattice inefficient.

Hilliard and Schwartz (2005)
- They match the first local moments of the lognormal jumps.
- Their lattice lacks the flexibility to suit derivatives’ specifications.
Main Results

- This talk proposes an efficient lattice model for the jump-diffusion process.
- The time complexity of our lattice is $O(n^{2.5})$.
- Our lattice is adjusted to suit the derivatives’ specification so that the price oscillation problem can be significantly suppressed.
Jump-Diffusion Process

- Define $S_t$ as the stock price at time $t$.
- Merton’s jump-diffusion model assumes that the stock price process can be expressed as
  \[ S_t = S_0 e^{(r - \lambda \bar{k} - 0.5 \sigma^2) t + \sigma z(t)} U(n(t)) \]  
  \[ V_t \equiv \ln \left( \frac{S_t}{S_0} \right) = X_t + Y_t, \]

- Decomposing Eq. (1) into the diffusion component and the jump component:
  \[ X_t \equiv \left( r - \lambda \bar{k} - 0.5 \sigma^2 \right) t + \sigma z(t) \]
  is a Brownian motion.
  \[ Y_t \equiv \sum_{i=0}^{n(t)} \ln (1 + k_i) \]
  is normal under Poisson compounding.

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The size of one time step is \( \Delta t = T/n \).

\( u, d, P_u, P_d \):

- Match the mean and variance of the stock return.
- \( ud = 1 \).
- \( P_u + P_d = 1 \).
Hilliard and Schwartz’s Lattice

- **Diffusion part** \((X_t)\)
  - Match mean and variance of \(X_{\Delta t}\).
  - Obtain \(P_u\) and \(P_d\).

- **Jump part** \((Y_t)\)
  - Match the first \(2m\) local moments of \(Y_{\Delta t}\).
  - Obtain \(q_j\) \((j = 0, \pm 1, \pm 2, \ldots, \pm m)\).
  - The node count of the lattice is \(O(n^3)\).
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Hilliard and Schwartz’s and Our Lattice

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Price Oscillation Problem

- Price oscillation problem is mainly due to the nonlinearity error.
- The solution of the nonlinearity error:
  - Making price level of the lattice coincide with the location where the option value function is highly nonlinear, such as the barriers and strike price.
Trinomial Structure

Theorem 1: the branching probabilities for the node $X$

$$P_u \alpha + P_m \beta + P_d \gamma = 0,$$
$$P_u (\alpha)^2 + P_m (\beta)^2 + P_d (\gamma)^2 = \text{Var},$$
$$P_u + P_m + P_d = 1.$$
Select $\Delta t$ to make $\frac{h'-l'}{2\sigma \sqrt{\Delta t}}$ be an integer.

Lay out the grid from barrier $L$ upward.

Automatically, barrier $H$ coincides with one level of nodes.

Obtain $P_u$, $P_m$, $P_d$ by Theorem 1 (p. 13).
Dealing with Jump Nodes

- Two phases: the diffusion phase and the jump phase.
- The node count of our lattice is $O(n^{2.5})$. 
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![Diagram of lattice construction with nodes labeled A to G and edges connecting them, illustrating the diffusion and jump phases.](image)
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**Time Complexity**

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*Figure:* time complexity.

\[
y = 2.50053 \times - 12.1905 \\
\text{Execution time of our lattice}
\]

\[
y = 3.06579 \times - 14.5049 \\
\text{Execution time of the HS lattice}
\]
Vanilla Options

Figure: Converge Property.
Barrier Options

Figure: Pricing a Single-Barrier Call Option.
This talk presents a novel, accurate, and efficient lattice model to price a huge variety of derivatives whose underlying asset follows the jump-diffusion process.

- It is the first attempt to reduce the time complexity of the lattice model for the jump-diffusion process to $O(n^{2.5})$.
- In contrast, that of previous work is $O(n^3)$.
- With the adjustable structure to fit derivatives' specifications, our lattice model make the pricing results converge smoothly.

According to the numerical results, our lattice model is superior to the existing methods in terms of accuracy, speed, and generality.